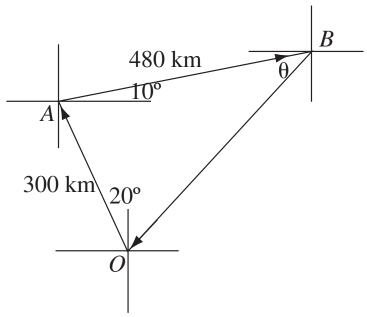
Chapter 4 • Vectors

Solutions for Selected Problems

Exercise 4.1

12. a.



The first leg of the trip is $\overrightarrow{OA}$. 240 km/h for $1\frac{1}{4}$ hours gives $|\overrightarrow{OA}| = 240 \times \frac{5}{4} = 300$ km.

The second leg of the trip is $\overrightarrow{AB}$ and $|\overrightarrow{AB}| = 240 \times 2 = 480$ km.

In ΔOAB, $\angle OAB = 70° + 10° = 80°$.

From the cosine law $|\overrightarrow{OB}|^2 = 300^2 + 480^2 - 2 \cdot 300 \cdot 480 \cos 80°$ $|\overrightarrow{OB}| \cong 519.99$.

Length of the third leg is 520 km.

Let $\angle ABO = \theta$ then from the sine law

$\frac{\sin\theta}{300} = \frac{\sin 80°}{|\overrightarrow{OB}|}$ $\quad \theta \cong 34.62°$.

The displacement vector for the third leg, $\overrightarrow{BO}$, has a magnitude of 520 km at *S* 45° *W*.

b. The total distance the aircraft travelled is $300 + 480 + 520 = 1300$ km. The time taken is $\frac{1300}{240} = 5.42$ hours or 5 hours 25 minutes.

13. Since $|(k-2)\vec{v}| < |4\vec{v}|$

$|k-2||\vec{v}| < |4||\vec{v}|$.

Since $\vec{v} \neq 0$, $|\vec{v}| \neq 0$ and $|k-2| < |4|$

$-4 < k - 2 < 4$

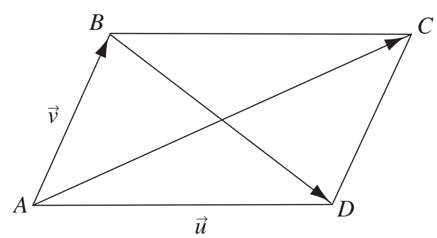
and $-2 < k < 6$.

14. If $\vec{u} = k\vec{v}$, $\vec{u}$ and $\vec{v}$ have the same direction for $k > 0$ and opposite directions for $k < 0$. In either case $\vec{u}$ will be parallel to $\vec{v}$.

If $\vec{u}$ is parallel to $\vec{v}$, $\vec{u}$ and $\vec{v}$ will have the same direction or opposite directions hence $\vec{u}$ is some multiple of $\vec{v}$ and $\vec{u} = k\vec{v}$.

Exercise 4.2

7. a.



In the diagram $\vec{u} + \vec{v} = \overrightarrow{AC}$ and

$\vec{u} - \vec{v} = \overrightarrow{BD}$.

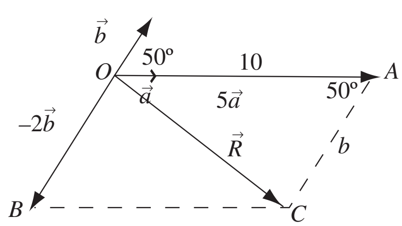
$\overrightarrow{AC}$ and $\overrightarrow{BD}$ are the diagonals of $\|^{gm}$ *ABCD*.

Now if $|\vec{u} + \vec{v}| = |\vec{u} - \vec{v}|$, the diagonals of the parallelogram are equal, hence the parallelogram is a rectangle and $\vec{u} \perp \vec{v}$.

b. If $|\vec{u} + \vec{v}| > |\vec{u} - \vec{v}|$, the angle between $\vec{u}$ and $\vec{v}$ will be acute.

c. If $|\vec{u} + \vec{v}| < |\vec{u} - \vec{v}|$, the angle between $\vec{u}$ and $\vec{v}$ will be obtuse.

9.



Let $\vec{R} = 5\vec{a} - 2\vec{b}$. Since $|\vec{a}| = 2$ and $|\vec{b}| = 3$, $|5\vec{a}| = 10$ and $|2\vec{b}| = 6$. From the cosine law

$|\vec{R}|^2 = 10^2 + 6^2 - 2 \cdot 10 \cdot 6 \cos 50°$

$|\vec{R}| \cong 7.672$

$\angle AOC = \theta$ then from the sine law

$\frac{\sin\theta}{6} - \frac{\sin 50°}{|\vec{R}|}$, $\theta \cong 36.80°$

$5\vec{a} - 2\vec{b}$ has a magnitude of 7.7 and makes an angle of 37° with $5\vec{a}$ and 93° with $-2\vec{b}$.

12. $\vec{a} = 3\vec{x} + 2\vec{y}$ ①

$\vec{b} = 5\vec{x} - 4\vec{y}$ ②

Solving the two equations for $\vec{x}$ and $\vec{y}$

① × 2 + ②

$11\vec{x} = 2\vec{a} + \vec{b}$

$\vec{x} = \frac{2}{11}\vec{a} + \frac{1}{11}\vec{b}.$

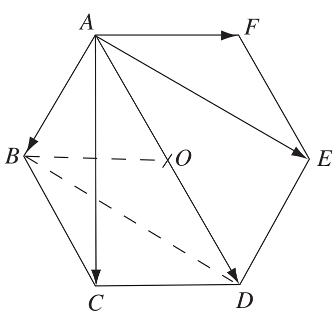
Substitute into ① $\vec{a} = \frac{6}{11}\vec{a} + \frac{3}{11}\vec{b} + 2\vec{y}$

$11\vec{a} = 6\vec{a} + 3\vec{b} + 22\vec{y}$

$\vec{y} = \frac{5}{22}\vec{a} - \frac{3}{22}\vec{b}.$

16. Since $\vec{a}$ and $\vec{b}$ form the sides of a parallelogram and since $|\vec{a}| = |\vec{b}|$, the parallelogram will be a rhombus. $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ will be the diagonals of this rhombus and since the diagonals of a rhombus are perpendicular to each other so will $(\vec{a} + \vec{b})$ be perpendicular to $(\vec{a} - \vec{b})$.

17.



$\overrightarrow{AB} = \overrightarrow{ED}$. Then $ABDE$ is a parallelogram and $\overrightarrow{AB} + \overrightarrow{AE} = \overrightarrow{AD}$.

$\overrightarrow{AF} = \overrightarrow{CD}$. Hence $ACDE$ is a parallelogram and $\overrightarrow{AC} + \overrightarrow{AF} = \overrightarrow{AD}$.

Now $\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{AD} + \overrightarrow{AE} + \overrightarrow{AF}$

$= \overrightarrow{AB} + \overrightarrow{AE} + \overrightarrow{AC} + \overrightarrow{AF} + \overrightarrow{AD}$

$= \overrightarrow{AD} + \overrightarrow{AD} + \overrightarrow{AD}$

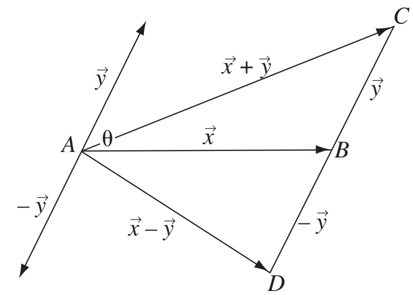
$= 3\overrightarrow{AD}.$

But O is he midpoint of AD and ΔABO is equilateral therefore $|\overrightarrow{AO}| = 1$ and $|\overrightarrow{AD}| = 2$.

Now $|\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{AD} + \overrightarrow{AE} + \overrightarrow{AF}| = 3|\overrightarrow{AD}|$
$= 6.$

Since the sum is equal to $3\overrightarrow{AD}$, the direction of the sum is along $\overrightarrow{AD}$ which makes an angle of 60° with $\overrightarrow{AF}$ and 60° with $\overrightarrow{AB}$.

18.



In ΔABC, $\angle ABC = 180° - \theta$ and from the cosine law $|x + y|^2 = |\vec{x}|^2 + |\vec{y}|^2 - 2|\vec{x}||\vec{y}| \cos(180° - \theta)$.

But $\cos(180° - \theta) = -\cos\theta$.

Hence $|\vec{x} + \vec{y}|^2 = |\vec{x}|^2 + |\vec{y}|^2 + 2|\vec{x}||\vec{y}| \cos\theta$. ①

From ΔABD, $\angle ABD = \theta$ and from the cosine law

$|\vec{x} - \vec{y}|^2 = |\vec{x}|^2 + |\vec{y}|^2 - 2|\vec{x}||\vec{y}| \cos\theta$. ②

Adding ① and ②

$|\vec{x} + \vec{y}|^2 + |\vec{x} - \vec{y}|^2 = 2|\vec{x}|^2 + 2|\vec{y}|^2.$

Now $|\vec{x}| = 11$, $|\vec{y}| = 23$, $|\vec{x} - \vec{y}| = 30$.

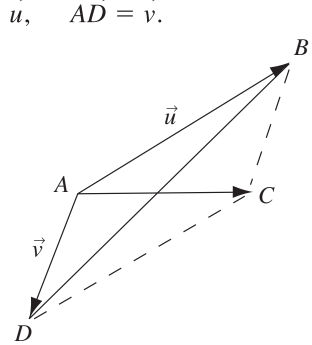
Hence $|\vec{x} + \vec{y}|^2 + 30^2 = 2 \cdot 11^2 + 2 \cdot 23^2$ $|\vec{x} + \vec{y}| = 20$.

19. Vectors $\vec{u} + \vec{v}$ and $\vec{u} - \vec{v}$ are the diagonals of a parallelogram where $\vec{u}$ and $\vec{v}$ are adjacent sides. Since $|\vec{u} + \vec{v}| < |\vec{u} - \vec{v}|$ the angle between $\vec{u}$ and $\vec{v}$ will be obtuse. Since the diagonals of a parallelogram bisect each other, draw $\vec{u} + \vec{v}$ and $\vec{u} - \vec{v}$ bisecting each other.

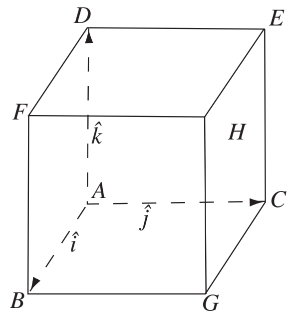
$\overrightarrow{AC} = \vec{u} + \vec{v}$

$\overrightarrow{DB} = \vec{u} - \vec{v}$

with $\overrightarrow{AB} = \vec{u}$, $\overrightarrow{AD} = \vec{v}$.



20.



a. $\overrightarrow{FG} = \overrightarrow{FH} + \overrightarrow{HG}$
$= \overrightarrow{AC} + \overrightarrow{DA}$
$= \hat{j} - \hat{k}.$

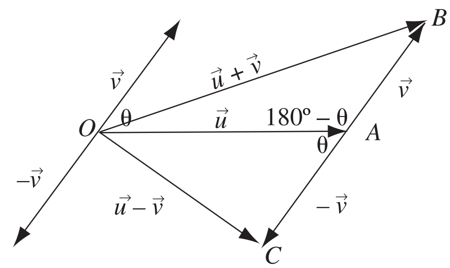
b. $\overrightarrow{BH} = \overrightarrow{BG} + \overrightarrow{GH}$
$= \hat{j} + \hat{k}$
$\overrightarrow{DH} = \overrightarrow{DE} + \overrightarrow{EH}$
$= \hat{j} + \hat{i}$
$\overrightarrow{CH} = \overrightarrow{CG} + \overrightarrow{GH}$
$= \hat{i} + \hat{k}$
$\overrightarrow{FE} = \overrightarrow{FH} + \overrightarrow{HE}$
$= \hat{j} - \hat{i}$
$\overrightarrow{EG} = \overrightarrow{EH} + \overrightarrow{HG}$
$= \hat{i} - \hat{k}.$

c. $\overrightarrow{BD} = \overrightarrow{BG} + \overrightarrow{GC} + \overrightarrow{CE}$
$= \hat{j} - \hat{i} + \hat{k}$
$\overrightarrow{BE} = -\hat{i} + \hat{j} + \hat{k}.$

d. $\overrightarrow{AH} = \hat{i} + \hat{j} + \hat{k}$
$\overrightarrow{CF} = \hat{i} - \hat{j} + \hat{k}$
$\overrightarrow{GD} = -\hat{i} - \hat{j} + \hat{k}.$

e. Face diagonal is $\overrightarrow{FG}$.
$|\overrightarrow{FG}| = \sqrt{1 + 1}$
$= \sqrt{2}.$
Body diagonal is $\overrightarrow{AH}$.
$|\overrightarrow{AH}| = \sqrt{1 + 1 + 1}$
$= \sqrt{3}.$

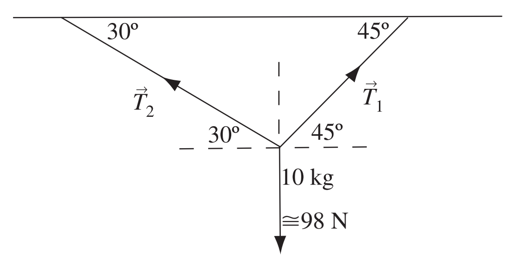
21.

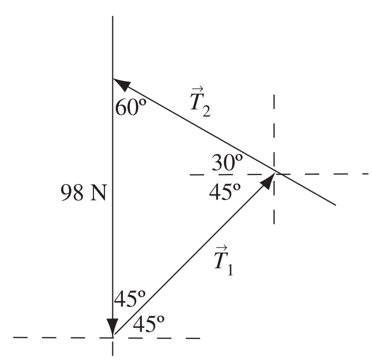


In ΔOAB, $\angle OAB = 180° - \theta$
and $\cos(180° - \theta) = -\cos\theta$.
From the cosine law
$|\vec{u} + \vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}||\vec{v}|\cos(180° - \theta)$
$|\vec{u} + \vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2 + 2|\vec{u}||\vec{v}|\cos\theta.$ ①
In ΔOAC, $\overrightarrow{AC} = -\vec{v}$, $|\overrightarrow{AC}| = |-\vec{v}| = |\vec{v}|$, $\angle OAC = \theta$.
From the cosine law
$|\vec{u} - \vec{v}|^2 = |\vec{u}|^2 + |-\vec{v}|^2 - 2|\vec{u}||-\vec{v}|\cos\theta$
$|\vec{u} - \vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}||\vec{v}|\cos\theta.$ ②
Adding ① and ②
$|\vec{u} + \vec{v}|^2 + |\vec{u} - \vec{v}|^2 = 2|\vec{u}|^2 + 2|\vec{v}|^2.$
$|\vec{u} + \vec{v}|^2 + |\vec{u} - \vec{v}|^2 = 2(|\vec{u}|^2 + |\vec{v}|^2).$

Exercise 4.3

9.





Represent the tensions in the cords by $\vec{T}_1$ and $\vec{T}_2$ as shown in the diagram.

From the force diagram and the sine law,

$$\frac{|\vec{T}_1|}{\sin 60°} = \frac{|\vec{T}_2|}{\sin 45°} = \frac{98}{\sin 75°}$$

$$|\vec{T}_1| = \frac{98 \sin 60°}{\sin 75°}$$

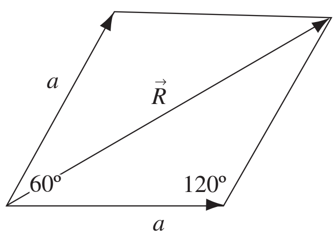
$$|\vec{T}_2| = \frac{98 \sin 45°}{\sin 75°}$$

$$|\vec{T}_1| \cong 87.9$$

$$|\vec{T}_2| \cong 71.7.$$

The tension in the cord making an angle of 45° with the ceiling is approximately 87.9 N and the tension in the other cord is approximately 71.7 N.

10.



Represent the magnitude of the forces by a.
From the cosine law we have

$$|\vec{R}|^2 = a^2 + a^2 - 2 \cdot a \cdot a \cos 120°$$

$$2a^2 + 2a^2 \cdot \frac{1}{2} = 30^2$$

$$3a^2 = 900$$

$$a^2 = 300$$

$$a = 10\sqrt{3}.$$

The magnitude of each force is $10\sqrt{3}$ N.

11. An object will be in a state of equilibrium when the resultant of all the forces acting on it is zero. This means that the sum of any two magnitudes must be greater than or equal to the magnitude of the third force.

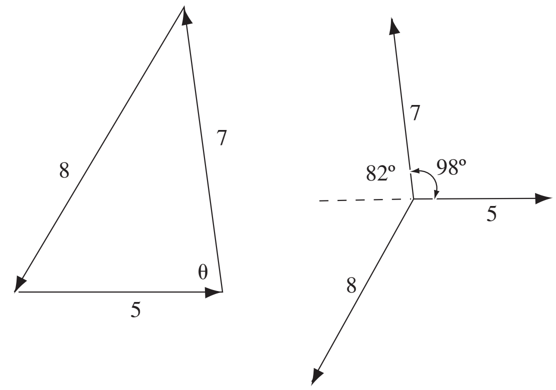
a. Since $5 + 2 = 7 < 13$, equilibrium cannot be achieved.

b. 7 N, 5 N, and 5 N can be arranged to produce equilibrium.

c. $13 + 14 = 17$, hence equilibrium can be achieved. In this case the three forces would be collinear.

d. Since $12 + 13 = 24 < 26$, equilibrium cannot be achieved.

12. a.



b. From the triangle of forces and the cosine law

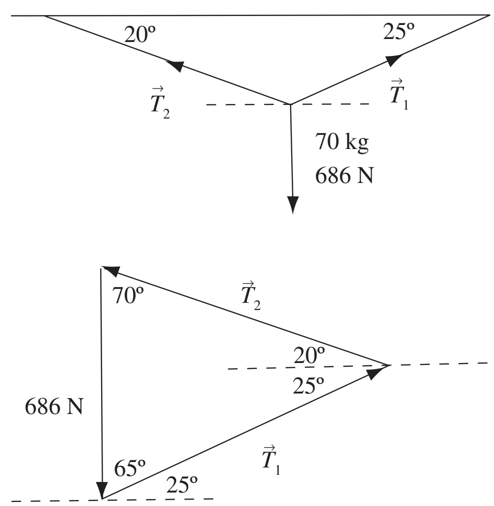
$$8^2 = 5^2 + 7^2 - 2 \cdot 5 \cdot 7 \cos \theta$$

$$\cos \theta = \frac{5^2 + 7^2 - 8^2}{2 \cdot 5 \cdot 7}$$

$$\theta \cong 82°.$$

The angle between the 5 N and 7 N forces will be $180° - 82° = 98°$.

13.



Represent the tensions in the cords by $\vec{T}_1$ and $\vec{T}_2$ as shown in the diagrams. From the triangle of forces and the sine law,

$$\frac{|\vec{T}_1|}{\sin 70°} = \frac{|\vec{T}_2|}{\sin 65°} = \frac{686}{\sin 45°}$$

$$|\vec{T}_1| = \frac{686 \sin 70°}{\sin 45°}$$

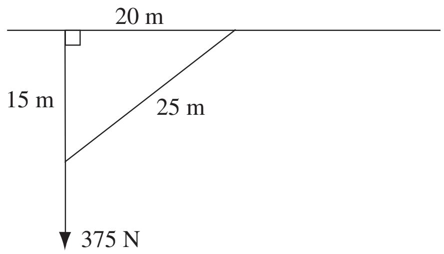
$$|\vec{T}_1| \cong 911.6$$

$$|\vec{T}_2| = \frac{686 \sin 65°}{\sin 45°}$$

$$|\vec{T}_2| \cong 879.3.$$

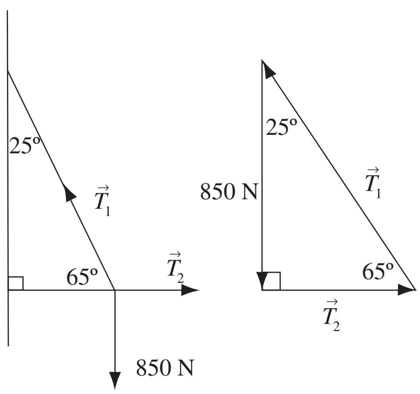
The tension in the rope making a 25° angle with the horizontal is approximately 911.6 N and in the other rope is approximately 879.3 N.

14.



The 20-, 15-, and 25- metre lengths form a right-angled triangle as shown in the diagram. Since the 375 N force is collinear with the 15 m steel wire, it will have a tension of 375 N and the tension along the 25 m steel wire will be 0 N.

15.



Let $\vec{T}_1$ represent the tension in the wire and $\vec{T}_2$ the compression in the steel brace as in the diagrams.

Now $\sin 65° = \dfrac{850}{|\vec{T}_1|}$

$|\vec{T}_1| = \dfrac{850}{\sin 65°}$

$|\vec{T}_1| \cong 937.9$

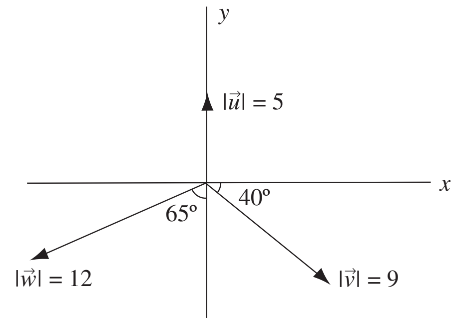
$\tan 65° = \dfrac{850}{|\vec{T}_2|}$

$|\vec{T}_2| = \dfrac{850}{\tan 65°}$

$|\vec{T}_2| \cong 396.4.$

The tension in the wire is approximately 937.9 N and the compression in the steel brace is approximately 396.4 N.

16.



Let $|\vec{p}_x|$ and $|\vec{p}_y|$ represent the components of $\vec{p}$ along the x-axis and y-axis respectively.

Now $|\vec{u}_y| = 5$, $|\vec{u}_x| = 0$

$|\vec{v}_x| = 9 \cos 40° \cong 6.9$

$|\vec{v}_y| = -9 \sin 40° \cong -5.8$

$|\vec{w}_x| = -12 \sin 65° \cong 10.9$

$|\vec{w}_y| = -12 \cos 65° \cong -5.1.$

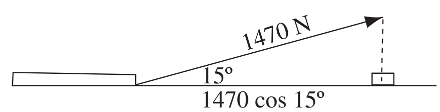
If $\vec{p} = \vec{w} + \vec{v} + \vec{u}$ then $|\vec{p}_x| = 12 \sin 65° - 9 \cos 40°$

$\cong 3.98$

and $|\vec{p}_y| = -9 \sin 40° - 12 \cos 65° + 5$

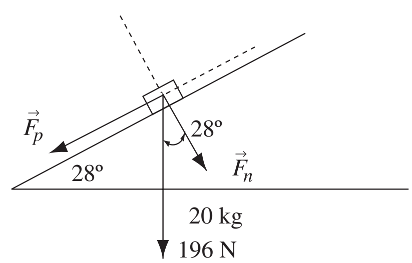
$\cong -10.86.$

17.



The horizontal component moving the log is 1470 cos 15° N ≅ 1420 N.

18. a.



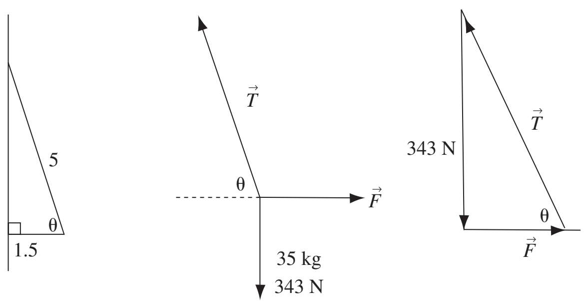
Let $|\vec{F}_p|$ and $|\vec{F}_n|$ represent the parallel and perpendicular components respectively.

$|\vec{F}_p| = 196 \sin 28°$ $\cong 92$ $\quad$ $|\vec{F}_n| = 196 \cos 28°$ $\cong 173$

The component parallel to the plane is 92 N and perpendicular to the plan 173 N.

b. The component normal to the ramp pushes down against the ramp and it in turn pushes back with an equal but opposite force. The component parallel to and down the ramp contributes to the luggage sliding down the ramp. If $|\vec{F}_p|$ is greater than the force of friction opposing $\vec{F}_p$, then the luggage will slide down the ramp.

19.



Let $\vec{F}$ represent the horizontal force and $\vec{T}$ the tension in the rope. Θ is the angle the rope makes with the horizontal.

$$\text{Now } \cos\theta = \frac{1.5}{5}$$
$$\theta \cong 72.54°$$

$$\sin\theta = \frac{343}{|\vec{T}|}$$

$$|\vec{T}| \cong \frac{343}{\sin 72.54°}$$

$$|\vec{T}| \cong 359.6$$

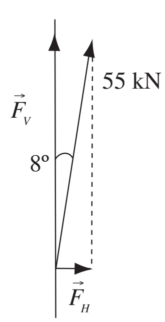
$$\tan\theta = \frac{343}{|\vec{F}|}$$

$$|\vec{F}| \cong \frac{343}{\tan 72.54°}$$

$$|\vec{F}| \cong 107.9.$$

A force of 107.9 N will hold the girl in this position and the tension in the rope is 359.6 N.

20. a.

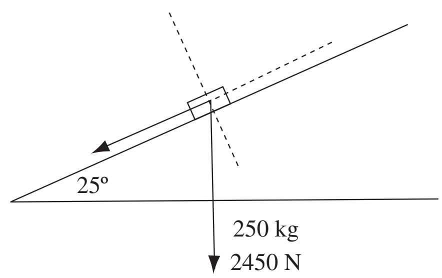


Let $|\vec{F}_v|$ represent the vertical component and $|\vec{F}_H|$ be the horizontal component.

$$\text{Now } |\vec{F}_v| = 66 \cos 8°$$
$$\cong 54.5$$
$$|\vec{F}_H| = 55 \sin 8°$$
$$\cong 7.7.$$

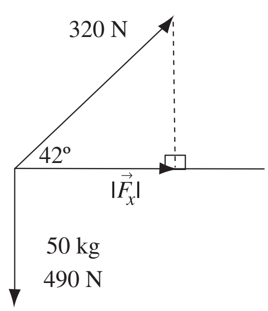
b. The vertical component is approximately 54.5 kN and is the component that gives the helicopter lift. The horizontal component is approximately 7.7 kN and is the component that moves the helicopter in a horizontal plane.

21.



The component that is parallel to the ramp is 2450 sin 25° ≅ 1035.4. The force of friction, to oppose this, must have a magnitude of at least 1035.4 N.

22.

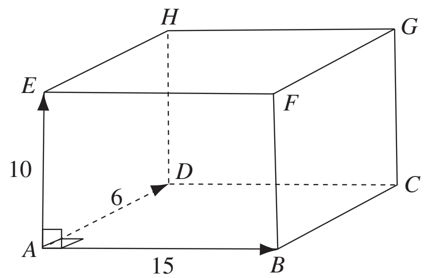


If $|\vec{F}_x|$ is the horizontal component then

$$|\vec{F}_x| = 320 \cos 42°$$
$$\cong 237.8.$$

The horizontal component causing the roller to move is approximately 238 N.

23.



Since the forces are perpendicular to each other, consider them acting along the edges of a rectangular solid with dimension 15 by 10 by 6. Now the magnitudes of the forces $|\vec{AD}| = 6$ N, $|\vec{AE}| = 10$ N, and $|\vec{AB}| = 15$ N. $\vec{AG}$ will be the sum of these forces

where $|\vec{AG}|^2 = 6^2 + 10^2 + 15^2$

$$|\vec{AG}| = 19.$$

In ΔAGB, $\angle GAB = \alpha$

$$\text{and } \cos\alpha = \frac{15}{19}$$

$$\alpha \cong 38°.$$

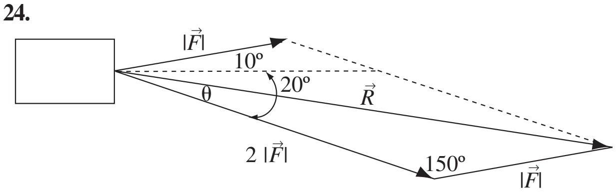
In ΔAEG, $\angle EAG = \beta$

$$\text{and } \cos\beta = \frac{10}{19}$$

$$\beta \cong 58°.$$

The magnitude of the resultant is 19 N and it makes angles of approximately 58° and 38° with the 10 N and 15 N forces respectively.

24.



Let $\vec{R}$ represent the vector along which the ship moves. From the parallelogram and cosine law, we

have $|\vec{R}|^2 = |\vec{F}|^2 + 4|\vec{F}|^2 - 2 \cdot |\vec{F}| \cdot 2|\vec{F}| \cos 150°$

$$= 5|\vec{F}|^2 + 2\sqrt{3}|\vec{F}|^2$$

$$|\vec{R}| = \sqrt{5 + 2\sqrt{3}}\,|\vec{F}|$$

From the sine law

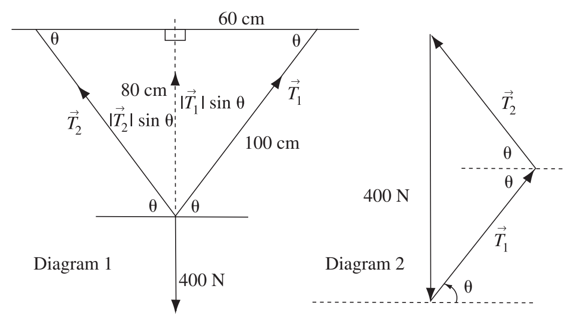
$$\frac{\sin\theta}{|\vec{F}|} = \frac{\sin 150°}{|\vec{R}|}$$

$$\sin\theta = \frac{\sin 150°}{\sqrt{5 + 2\sqrt{3}}}$$

$$\theta \cong 9.89°.$$

The ship will move approximately $20° - 9.89° \cong 10°$ off the starboard bow.

25. a.



Let $\vec{T}_1$ and $\vec{T}_2$ be the tension in each length of string. Since the mass is suspended from the midpoint of the cord, $|\vec{T}_1| = |\vec{T}_2|$. From diagram 1 the vertical components of $\vec{T}_1$ and $\vec{T}_2$ are $|\vec{T}_1| \sin\theta$ and $|\vec{T}_2| \sin\theta$. For equilibrium the sum of these vertical components will be 400.

Therefore $|\vec{T}_1| \sin\theta + |\vec{T}_2| \sin\theta = 400.$

But $|\vec{T}_1| = |\vec{T}_2|$ therefore $2|\vec{T}_1| \sin\theta = 400.$

From the 100, 80, 60 triangle, $\sin\theta = \frac{4}{5}$

hence $\frac{8}{5}|\vec{T}_1| = 400$

$$|\vec{T}_1| = 250.$$

The tension in each length would be 250 N hence the string will support the weight.

OR

From diagram 2 and the sine law

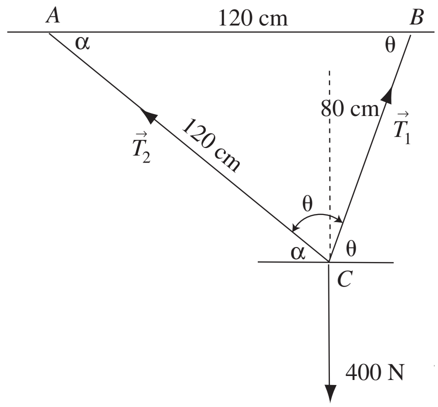
$$\frac{|\vec{T}_1|}{\sin(90° - \theta)} = \frac{400}{\sin 2\theta}.$$

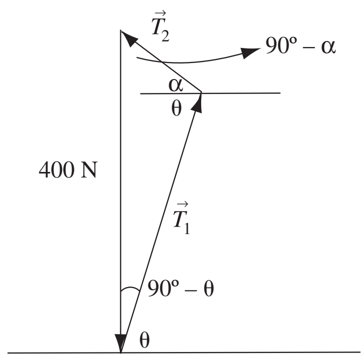
But $\sin(90° - \theta) = \cos\theta$, $\sin 2\theta = 2\sin\theta\cos\theta$ and $\sin\theta = \frac{4}{5}$.

Hence $|\vec{T}_1| = \frac{400\cos\theta}{2\sin\theta\cos\theta} = \frac{200}{\sin\theta} = 250.$

Conclusion as above.

b.





Represent the tensions as $\vec{T}_1$ and $\vec{T}_2$ and the angles in ΔABC and α and θ as shown in the diagram.
Since $AC = AB = 120$, $\angle ACB = \angle ABC = \theta$.
From the cosine law

$$80^2 = 120^2 + 120^2 - 2 \cdot 120 \cdot 120 \cos \alpha$$

$$\cos \alpha = \frac{2 \cdot 120^2 - 80^2}{2 \cdot 120 \cdot 120}$$

$$\alpha \cong 38.94°.$$

Also $\alpha + 2\theta = 180°$

therefore $\theta \cong 70.53°$.

From the sine law

$$\frac{|\vec{T}_1|}{\sin (90° - \alpha)} = \frac{|\vec{T}_2|}{\sin (90° - \theta)} = \frac{400}{\sin (\alpha - \theta)}$$

$$90° - \alpha = 51.06°$$
$$90° - \theta = 19.47°$$
$$\alpha + \theta = 109.47°$$

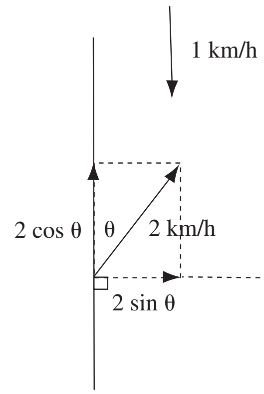
$$|\vec{T}_1| \cong \frac{400 \sin 51.06°}{\sin 109.47°} \cong 330.0$$

$$|\vec{T}_2| \cong \frac{400 \sin 19.47°}{\sin 109.47°} \cong 141.4.$$

Since the tension $|\vec{T}_1| \cong 330$ N > 300 N, the string will not support the 400 N weight.

Exercise 4.4

2. a.



Let the angle to the bank be θ. The component perpendicular to the bank will be 2 sin θ, the speed that takes him across the river, and the component parallel to the bank is 2 cos θ. For the man to swim directly across the river then

$$2 \cos \theta = 1$$

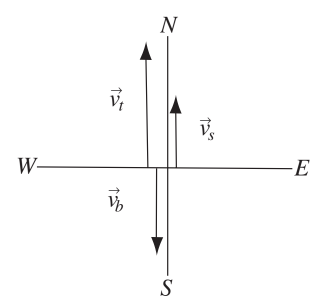
$$\cos \theta = \frac{1}{2}$$

and $\theta = 60°$.

The man must swim at an angle of 60° to the bank if he is to reach a point directly across from his starting point.

b. If the speed of the current if 4 km/h, $2 \cos \theta = 4$, $\cos \theta = 2$ which is not possible since $|\cos \theta| \leq 1$. He will not be able to swim to a point directly across the river in this case. As long as the current is less than 2 km/h, he will be able to swim to a point directly across the river.

3.



Let $\vec{v}_s$, $\vec{v}_b$, and $\vec{v}_t$ represent the velocities of the streetcar, bus, and taxi respectively and $\vec{v}_s = 35$, $\vec{v}_b = -42$, $\vec{v}_t = 50$ where north is positive and south is negative.

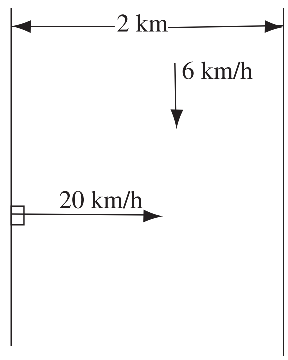
a. The velocity of the streetcar relative to the taxi, $\vec{v}_s - \vec{v}_t = 35 - (+50) = -15$, is 15 km/h south.

b. The velocity of the streetcar relative to the bus, $\vec{v}_s - \vec{v}_b = 35 - (-42) = 77$, is 77 km/h north.

c. The velocity of the taxi relative to the bus, $\vec{v}_t - \vec{v}_b = 50 - (-42) = 92$, is 92 km/h north.

d. The velocity of the bus relative to the streetcar, $\vec{v}_b - \vec{v}_s = -42 - (35) = -77$, is 77 km/h south.

4. a.

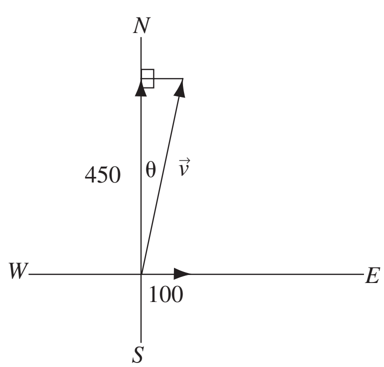


The distance downstream will be the distance travelled in 6 min at 6 km/h, $\frac{1}{10} \times 6 = 0.6$, 0.6 km.

He will touch the bank 0.6 km downstream from the marina and will be there in 6 minutes.

b. The boat will proceed across the river at a speed of 20 km/h regardless of the speed of the current. Hence the time it takes to cross the river will be the time it takes to travel 2 km at 20 km/h,
$\frac{2}{20} = \frac{1}{10}$, 6 minutes.

5.



Let the resultant velocity be $\vec{v}$.

Now $|\vec{v}|^2 = 450^2 + 100^2$

$|\vec{v}| \cong 460.9772$

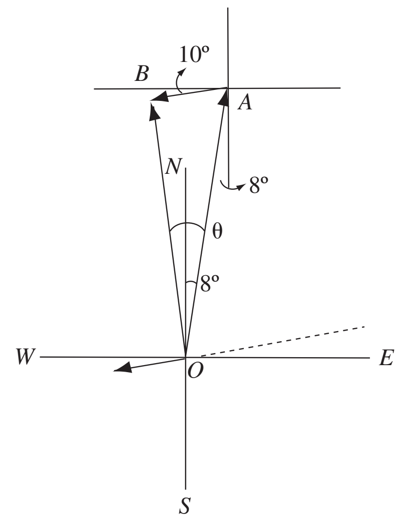
and $\tan\theta = \frac{100}{450}$

$\theta \cong 12.53°$.

a. The plane will travel a distance of $3 \times |\vec{v}| \cong 1383$ km in 3 hours.

b. The direction of the plane is approximately $N\ 13°\ E$.

6.



Adding the vectors creates ΔOAB where $\overrightarrow{OA}$ is the velocity of the aircraft, $|\overrightarrow{OA}| = 175$, $\overrightarrow{AB}$ is the velocity of the wind, $|\overrightarrow{AB}| = 40$, and $\angle BAO = 90° - 10° - 8°$
$= 72°$.

$\overrightarrow{OB} = \vec{v}$ is the resultant velocity and $\angle BOA = \theta$.

From the cosine law

$|\vec{v}|^2 = 40^2 + 175^2 - 2 \cdot 40 \cdot 175 \cos 72°$

$|\vec{v}| \cong 167.03$.

From the sine law

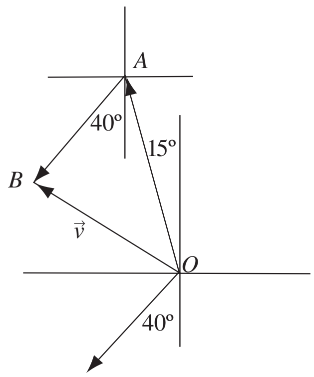
$$\frac{\sin\theta}{40} = \frac{\sin 72°}{|\vec{v}|}$$

$$\sin\theta = \frac{40 \sin 72°}{|\vec{v}|}$$

$$\theta \cong 13.17°.$$

The ground velocity is approximately 167 km/h in a direction $N\ 5°\ W$ $(13° - 8° = 5°)$.

7.



Adding the vectors creates ΔOAB where $\overrightarrow{OA}$ is the boat's velocity, $|\overrightarrow{OA}| = 3$; $\overrightarrow{AB}$ is the current's velocity, $|\overrightarrow{AB}| = 2$, and $\angle BAO = 55°$, $\angle BOA = \theta$.
From the cosine law

$|\vec{v}|^2 = 3^2 + 2^2 - 2 \cdot 3 \cdot 2 \cos 55°$

$|\vec{v}| \cong 2.473.$

From the sine law

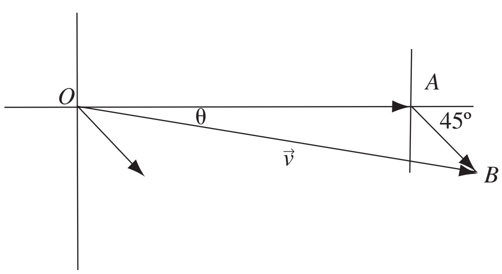
$$\frac{\sin \theta}{2} = \frac{\sin 55°}{|\vec{v}|}$$

$$\sin \theta = \frac{2 \sin 55°}{|\vec{v}|}$$

$\theta \cong 41.48°.$

The velocity is approximately 2.5 m/s in a direction of $N\ 56°\ W$ ($41.48° + 15° = 56.48°$).

8.



Adding the vectors forms ΔOAB where the plane steering east at 240 km/h is represented by $\overrightarrow{OA}$, the wind from the northwest is represented by $\overrightarrow{AB}$, $|\overrightarrow{AB}| = 65$, and the plane's actual velocity is $\vec{v}$ where $\angle AOB = \theta$.
From the cosine law

$|\vec{v}|^2 = 240^2 + 65^2 - 2 \cdot 240 \cdot 65 \cos 135°$

$|\vec{v}| \cong 289.63.$

From the sine law

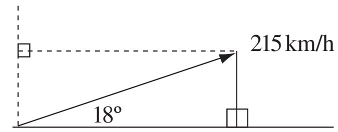
$$\frac{\sin \theta}{65} = \frac{\sin 135°}{|\vec{v}|}$$

$$\sin \theta = \frac{65 \sin 135°}{|\vec{v}|}$$

$\theta \cong 9.1°.$

The plane's actual direction is approximately $S\ 81°\ E$.

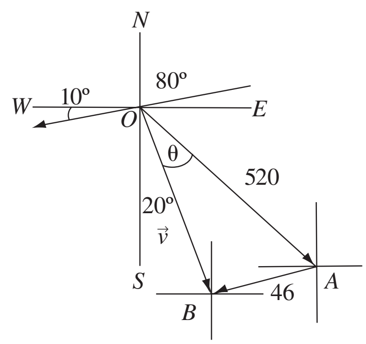
9.



a. The horizontal component is $215 \cos 18° \cong 204$ km/h. The vertical component is $215 \sin 18° \cong 66$ km/h.

b. The horizontal component is the speed that the jet advances. The vertical component is the speed at which the jet gains vertical altitude.

10.



$\overrightarrow{OA}$ represents the vector along which the plane steers. $|\overrightarrow{OA}| = 520$ km/h, $\angle BOA = \theta$ hence the plane steers at $S\ (20 + \theta)°\ E$. $\overrightarrow{AB}$ represents the wind velocity, $|\overrightarrow{AB}| = 46$ km/h and $\angle ABO = 80° + 20° = 100°$. $\overrightarrow{OB} = \vec{v}$ represents the velocity with respect to the ground.
From the sine law

$$\frac{\sin \theta}{46} = \frac{\sin 100°}{520} = \frac{\sin \angle OAB}{|\vec{v}|}$$

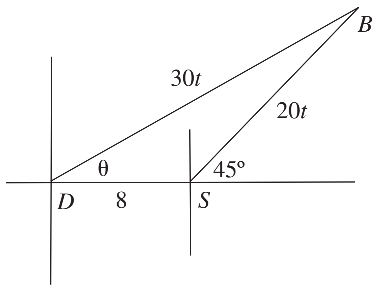
$$\sin \theta = \frac{46 \sin 100°}{520}$$

$\theta \cong 4.997°.$

$\angle OAB = 180° - \theta - 100°$
$\cong 75°$
$\therefore |\vec{v}| = \frac{520 \sin \angle OAB}{\sin 100°}$
$|\vec{v}| \cong 510.04.$

The pilot should steer in a direction $S\ 25°\ E$ and the plane's ground speed will be approximately 510 km/h.

11.



The destroyer travels in a direction θ as in the diagram and will intercept the sub in t hours. Hence the distance $DB = 30t$ nautical miles and $SB = 20t$ nautical miles. $\angle OSB = 135°$ and from the sine law

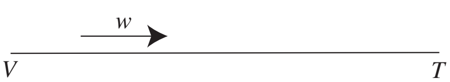
$$\frac{\sin \theta}{20t} = \frac{\sin 135°}{30t}$$

$$\sin \theta = \frac{20 \sin 135°}{30}$$

$\theta \cong 28°, 0 \le \theta \le 90°$

The destroyer should travel in a direction of $N\ 62°\ E$ to intercept the submarine.

12. a.



Represent the velocity of the aircraft as $\vec{v}$ and the wind velocity as $\vec{w}$, $v > w$. Let the distance between Toronto and Vancouver be x km. The speed in going from Vancouver to Toronto with the wind is $(v + w)$ km/h and from Toronto to Vancouver will be $(v - w)$ km/h. The time to go from Vancouver to Toronto will be $\frac{x}{v + w}$ h and from Toronto to Vancouver $\frac{x}{v - w}$ h.

Total time is $T_a = \frac{x}{v + w} + \frac{x}{v - w}$

$= x\left[\frac{v - w + v + w}{(v + w)(v - w)}\right]$

$= \frac{2xv}{v^2 - w^2}.$

b. When there is no wind, the time required to travel from Vancouver to Toronto is $\frac{x}{v}$ h and from Toronto to Vancouver is $\frac{x}{v}$ h.

Total time is $T_b = \frac{2x}{v}.$

Now $T_a - T_b$

$= \frac{2xv}{v^2 - w^2} - \frac{2x}{v}$

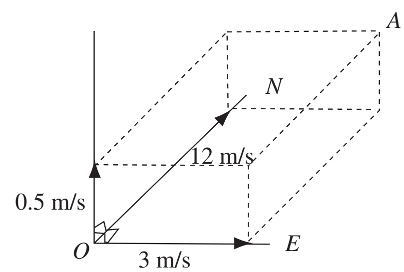
$= 2x\left[\frac{v^2 - (v^2 - w^2)}{v(v^2 - w^2)}\right]$

$= \frac{2xw^2}{v(v^2 - w^2)} > 0$

Therefore $T_a - T_b > 0$

Since $T_a - T_b > 0$, $T_a > T_b$, it takes longer to travel from Vancouver to Toronto and back when there is a wind.

13.

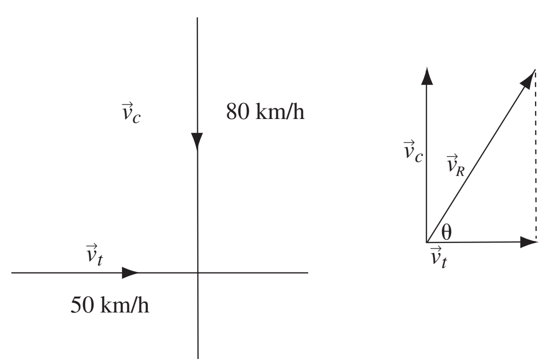


The speed relative to the ocean floor is represented by OA, a diagonal of a rectangular solid with sides of length 0.5, 3, and 12, as shown in the diagram.

$OA^2 = (0.5)^2 + 12^2 + 3^2$
$OA \cong 12.379$

The speed of the sailor relative to the ocean floor is approximately 12.4 m/s.

14. Let $\vec{v}_c$ represent the velocity of the car and $\vec{v}_t$ the velocity of the truck. Vector $\vec{v}_R$, the velocity of the truck relative to the car, is such that $\vec{v}_R = \vec{v}_t - \vec{v}_c$.



$|\vec{v}_R|^2 = 80^2 + 50^2$

$|\vec{v}_R| \cong 94.34$

θ is the angle between $\vec{v}_R$ and $\vec{v}_t$.

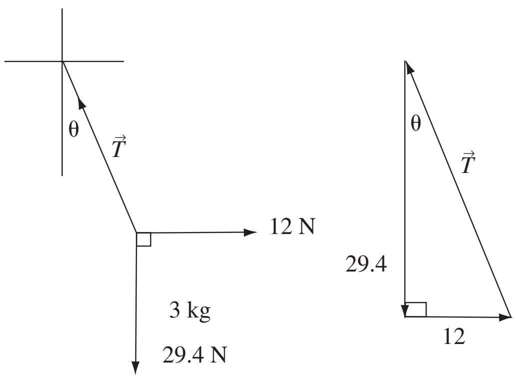
$\tan\theta = \dfrac{80}{50}$

$\theta \cong 58°.$

The velocity of the truck relative to the car is approximately 94.3 km/h in a direction N 32° E.

Review Exercise

7.



Represent the tension in the string by $\vec{T}$ and the angle the string makes with the vertical by θ as shown in the diagrams.
Since the system is in equilibrium the sum of the three forces will be $\vec{O}$ as shown in the triangle diagram.

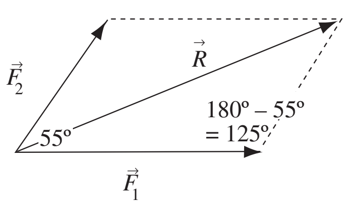
Now $|\vec{T}|^2 = 29.4^2 + 12^2$

$|\vec{T}| \cong 31.75$

$\tan\theta = \dfrac{12}{29.4}, \theta \cong 22.2°.$

The tension in the string has a magnitude of 32 N and the string makes an angle of 22° with the vertical.

8. a.



Represent the resultant by $\vec{R}$. From the cosine law

$|\vec{R}|^2 = |\vec{F}_1|^2 + |\vec{F}_2|^2 - 2|\vec{F}_1||\vec{F}_2| \cos 125°$

$= 54^2 + 34^2 - 2 \cdot 54 \cdot 34 \cos 125°$

$|\vec{R}| \cong 78.601.$

The magnitude of the resultant is approximately 79 N.

b. $|\vec{F}_1| = 21, |\vec{F}_2| = 45, \theta = 140°$

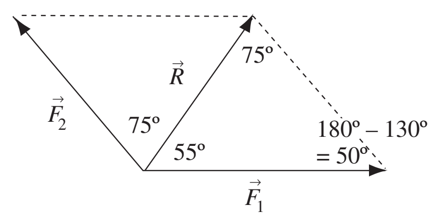
$|\vec{R}|^2 = |\vec{F}_1|^2 + |\vec{F}_2|^2 - 2|\vec{F}_1||\vec{F}_2| \cos 40°$

$= 21^2 + 45^2 - 2 \cdot 21 \cdot 45 \cos 40°$

$|\vec{R}| \cong 31.909$

The magnitude of the resultant is approximately 32 N.

9.



From the sine law

$$\frac{|\vec{R}|}{\sin 50°} = \frac{|\vec{F}_1|}{\sin 75°} = \frac{|\vec{F}_2|}{\sin 55°}$$

$|\vec{R}| = 480$ N

$|\vec{F}_1| = \dfrac{|\vec{R}| \sin 75°}{\sin 50°}$

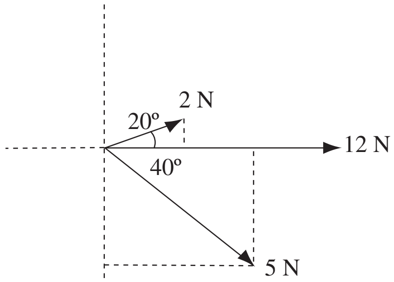
$|\vec{F}_1| \cong 605.2$

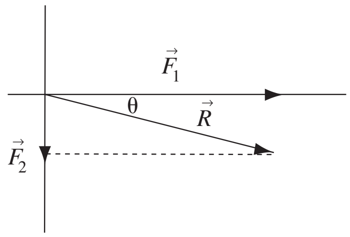
$|\vec{F}_2| = \dfrac{|\vec{R}| \sin 55°}{\sin 50°}$

$|\vec{F}_2| \cong 513.3.$

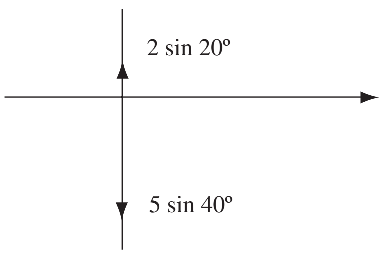
The magnitudes of the two forces are approximately 605 N and 513 N.

10.





Resolve the 2 N and 5 N forces into rectangular components along and perpendicular to the 12 N force.



Let $\vec{R}$ be the resultant of $\vec{F}_1$ and $\vec{F}_2$ where

$$|\vec{F}_1| = 12 + 2\cos 20° + 5\cos 40° \cong 17.7096$$

$$\text{and } |\vec{F}_2| = 5\sin 40° - 2\sin 20° \cong 2.5299.$$

$$\text{Now } |\vec{R}|^2 = |\vec{F}_1|^2 + |\vec{F}_2|^2$$

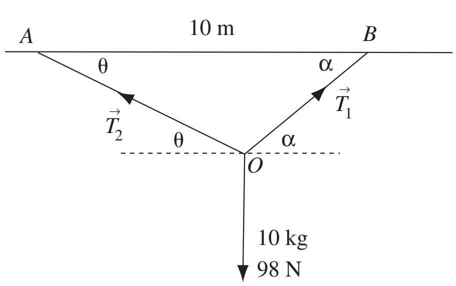
$$|\vec{R}| \cong 17.8894$$

$$\tan\theta = \frac{|\vec{F}_2|}{|\vec{F}_1|}$$

$$\theta \cong 8.13°.$$

The resultant has a magnitude of approximately 17.9 N and makes an angle of 8° with the 12 N force and 32° with the 5 N force.

11.



Let $\vec{T}_1$ and $\vec{T}_2$ represent the tensions in each string and α and θ be the angles that the strings make with the ceiling as shown in the diagram. In ΔOAB, $OA = 7$, $OB = 5$, $AB = 10$. From the cosine law

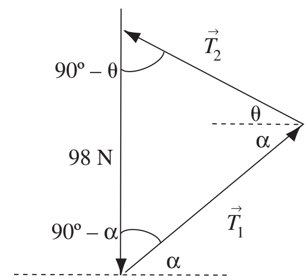
$$7^2 = 10^2 + 5^2 - 2\cdot 5\cdot 10\cos\alpha$$

$$\cos\alpha = \frac{10^2 + 5^2 - 7^2}{2\cdot 5\cdot 10}$$

$$\cos\alpha = \frac{19}{25}, \alpha \cong 40.5°.$$

$$\text{Also } 5^2 = 7^2 + 10^2 - 2\cdot 7\cdot 10\cos\theta$$

$$\cos\theta = \frac{31}{35}, \theta \cong 27.7°.$$



From the sine law

$$\frac{98}{\sin(\alpha + \theta)} = \frac{|\vec{T}_1|}{\sin(90° - \theta)} = \frac{|\vec{T}_2|}{\sin(90° - \alpha)}.$$

But $\sin(90° - \theta) = \cos\theta$, $\sin(90° - \alpha) = \cos\alpha$ and $\alpha + \theta = 68.2°$

$$|\vec{T}_1| = \frac{98\cos 27.7°}{\sin 68.2°}$$

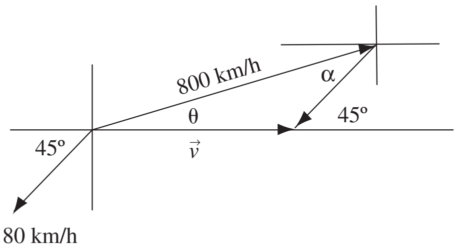
$$\cong 93.5$$

$$|\vec{T}_2| = \frac{98\cos 40.5°}{\sin 68.2°}$$

$$\cong 80.3.$$

The tension in the 5 m string is 93.5 N and in the 7 m string is 80.3 N.

12.



To fly due east let the bearing of the plane be θ north of east, and $\vec{v}$ represent the velocity due east. From the sine law

$$\frac{\sin\theta}{80} = \frac{\sin 135°}{800} = \frac{\sin\alpha}{|\vec{v}|}$$

$$\sin\theta = \frac{\sin 135°}{10}$$

$$\theta \cong 4.1°$$
$$\alpha = 180° - 135° - 4.1°$$
$$= 40.9°.$$

Now $\dfrac{\sin 40.9°}{|\vec{v}|} = \dfrac{\sin 135°}{800}$

$$|\vec{v}| = \frac{800 \sin 40.9°}{\sin 135°}$$

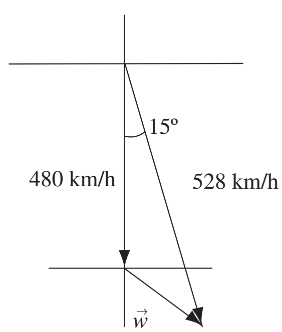
$$|\vec{v}| \cong 740.8.$$

a. The plane's heading should be *N* 85.9° *E*.

b. The time required to go 800 km at 740.8 km/h is

$$\frac{800}{740.8} \cong 1 \text{ hour } 5 \text{ min.}$$

13.



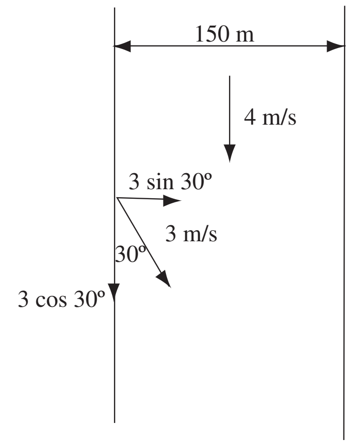
The wind will be from the north-west to push the plane on a flight of *S* 15° *E*. If $\vec{w}$ is the wind velocity, then from the cosine law

$$|\vec{w}|^2 = 480^2 + 528^2 - 2 \cdot 480 \cdot 528 \cos 15°$$

$$|\vec{w}| \cong 139.9$$

The wind speed is approximately 140 km/h.

14.

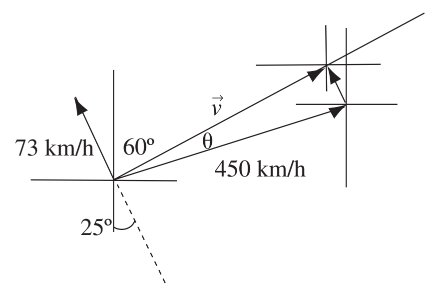


a. Resolve the velocity of 3 m/s into rectangular components, 3 cos 30° with the current and 3 sin 30° perpendicular to the current. Her speed downstream will be (4 + 3 cos 30°) m/s. Her distance downstream in 10 s will be 10(4 + 3 cos 30°) ≅ 65.98 m.

b. Her speed going across the river is 3 sin 30° m/s. The time required to go 150 m is

$$\frac{150}{3 \sin 30°} = 100 \text{ s.}$$

15. a.



Let $\vec{v}$ represent the ground velocity. From the triangle of vectors let θ be the angle between $\vec{v}$ and 450 km/h as shown. The angle opposite 450 is 60° + 25° = 85°. From the sine law

$$\frac{\sin\theta}{73} = \frac{\sin 85°}{450}$$

$$\sin\theta = \frac{73 \sin 85°}{450}$$

$$\theta \cong 9.3°$$

The pilot should steer on a heading of *N* 69° *E*.

b. $\dfrac{|\vec{v}|}{\sin 86°} = \dfrac{450}{\sin 85°}$

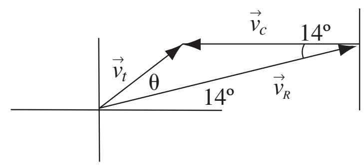
$$|\vec{v}| = \frac{450 \sin 86°}{\sin 85°}$$

$$|\vec{v}| \cong 450.62$$

The ground speed is approximately 451 km/h.

c. The time to fly 350 km is $\dfrac{350}{451} \cong 0.776$ hours or 47 min.

16.



Let $\vec{v}_R$ be the relative velocity of the tanker to the cutter.

$\vec{V}_t$ the velocity of the tanker

$\vec{V}_c$ the velocity of the cutter.

Now $\vec{v}_R = \vec{v}_t - \vec{v}_c$

$|\vec{v}_R| = 19$ knots, $\quad |\vec{v}_c| = 12$ knots.

From the cosine law

$$|\vec{v}_t|^2 = 12^2 + 19^2 - 2 \cdot 12 \cdot 19 \cos 14°$$

$$|\vec{v}_t| \cong 7.9.$$

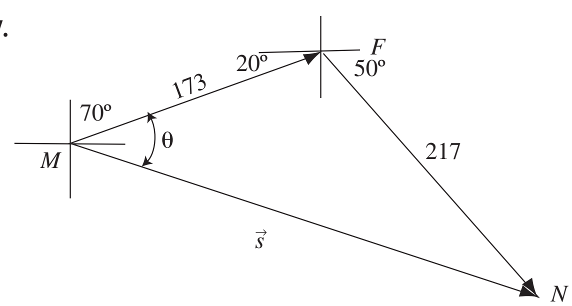
From the sine law

$$\frac{\sin \theta}{12} = \frac{\sin 14°}{|\vec{v}_t|}$$

$$\theta \cong 21.5°.$$

The actual velocity of the tanker is 7.9 knots on a bearing of N 54° E.

17.



Let the displacement vector be $\vec{s}$ and $\angle FMN$ be θ.
$\angle MFN = 110°$.

From the cosine law

$$|\vec{s}|^2 = 173^2 + 217^2 - 2 \cdot 173 \cdot 217 \cos 110°$$

$$|\vec{s}| \cong 320.46.$$

Now $\sin \theta = \dfrac{217 \sin 110°}{|\vec{s}|}$

$$\theta \cong 39.52°.$$

The displacement vector has a magnitude of 320 km with a bearing of S 70° E.

18. $a\vec{u} + b\vec{v} = \vec{O}$

$a\vec{u} = -b\vec{v}$

If $\vec{u}$ and $\vec{v}$ are not collinear then $a = b = 0$.

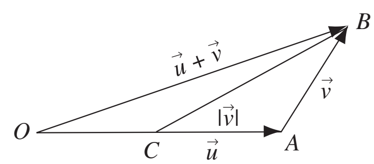
If $\vec{u}$ and $\vec{v}$ are collinear and have opposite directions then $|a\vec{u}| = |b\vec{v}|$.

$$|a||\vec{u}| = |\vec{b}||\vec{v}|$$

Let $\dfrac{|a|}{|\vec{v}|} = \dfrac{|b|}{|\vec{u}|} = k, \quad k \in R.$

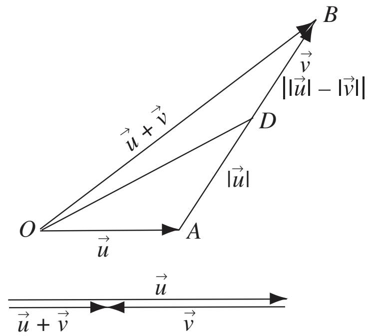
Now $a = k|\vec{v}|$ and $b = k|\vec{u}|$.

19. Case I.



$|\vec{u}| > |\vec{v}|$. In the diagram, let $\overrightarrow{OA} = \vec{u}$, $\overrightarrow{AB} = \vec{v}$. Therefore $\overrightarrow{OB} = \vec{u} + \vec{v}$. Locate C in OA so that $CA = |\vec{v}|$ hence $OC = |\vec{u}| - |\vec{v}|$.
In ΔCAB, $CA = AB$ therefore $\angle ACB = \angle ABC$ and each of these angles is obtuse. In ΔOCB, $\angle OCB$ is the largest angle, therefore OB is the longest side, hence $OC < OB$, i.e., $|\vec{u}| - |\vec{v}| < |\vec{u} + \vec{v}|$.

Case II.



$|\vec{u}| < |\vec{v}|$. Similar proof to the above (see diagram).

$AD = |\vec{u}|$

$DB = \left||\vec{u}| - |\vec{v}|\right|$

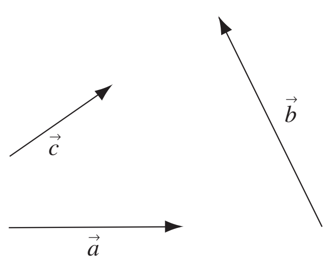
In either case, $\left||\vec{u}| - |\vec{v}|\right| < |\vec{u} + \vec{v}|$.

Equality holds if $\vec{u}$ is parallel to $\vec{v}$ but in the opposite direction or if $|\vec{u}| = |\vec{v}| = 0$ hence $\left||\vec{u}| - |\vec{v}|\right| \le |\vec{u} + \vec{v}|$.

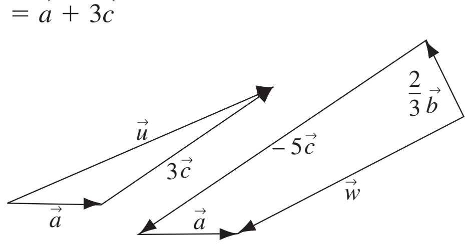
Chapter 4 Test

1. $|\vec{u} + \vec{v}| = |\vec{u}| + |\vec{v}|$ when $\vec{u}$ and $\vec{v}$ are collinear and have the same direction.

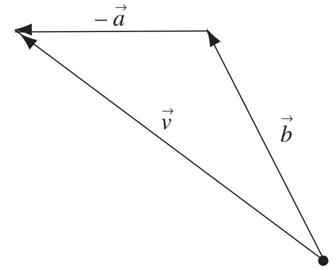
2.



a. $\vec{u} = \vec{a} + 3\vec{c}$



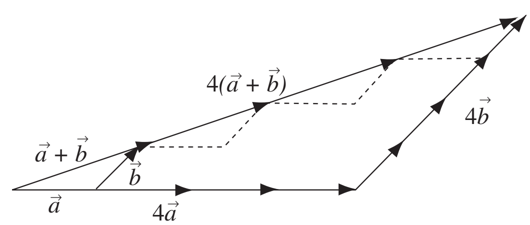
b. $\vec{v} = \vec{b} - \vec{a}$.



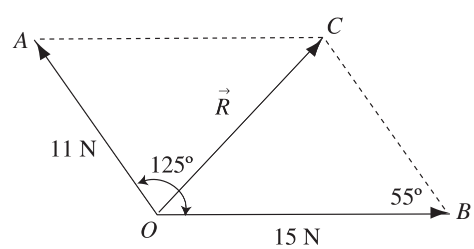
c. $\vec{w} = \frac{2}{3}\vec{b} - 5\vec{c} + \vec{a}$.

3. $3(4\vec{u} + \vec{v}) - 2\vec{u} - 3(\vec{u} - \vec{v})$
$= 12\vec{u} + 3\vec{v} - 2\vec{u} - 3\vec{u} + 3\vec{v}$
$= 7\vec{u} + 6\vec{v}$.

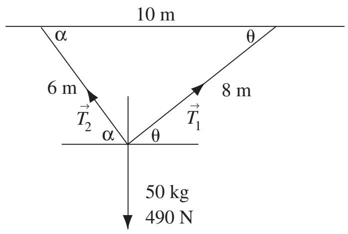
4. $4(\vec{a} + \vec{b}) = 4\vec{a} + 4\vec{b}$ is called the distributive property.



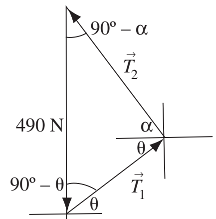
5. From the parallelogram law the resultant $\vec{R} = \overrightarrow{OC}$ and $\angle OBC = 55°$.
Now $|\vec{R}|^2 = 15^2 + 11^2 - 2 \cdot 15 \cdot 11 \cos 55°$
$|\vec{R}| \cong 12.5$.
The magnitude of the resultant is approximately 12.5 N.



6.



Position Diagram



Force Diagram

From the position diagram, the 6-m, 8-m, and 10-m lengths form a right triangle.

Therefore $\sin\theta = \frac{3}{5}$, $\sin\alpha = \frac{4}{5}$, and $\alpha + \theta = 90°$.

From the force diagram, $\alpha + \theta = 90°$

and $\sin(90° - \theta) = \frac{|\vec{T}_2|}{490}$

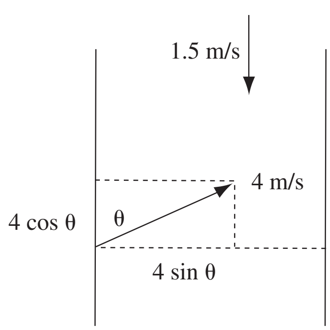
$$\begin{aligned} \text{therefore } |\vec{T}_2| &= 490 \sin\alpha \\ &= 490 \times \frac{4}{5} \\ |\vec{T}_2| &= 392 \end{aligned}$$

$\sin(90° - \alpha) = \frac{|\vec{T}_1|}{490}$

$$\begin{aligned} \text{therefore } |\vec{T}_1| &= 490 \sin\theta \\ &= 490 \times \frac{3}{5} \\ |\vec{T}_1| &= 294. \end{aligned}$$

The tension in each part of the cable is 294 N (8-m length) and 392 N (6-m length).

7.



Let the boat steer at an angle of θ to the bank as shown in the diagram. The component of the boat's speed against the current is 4 cos θ and the component perpendicular to the current is 4 sin θ. To go directly across the river, the component against the current must equal the current.

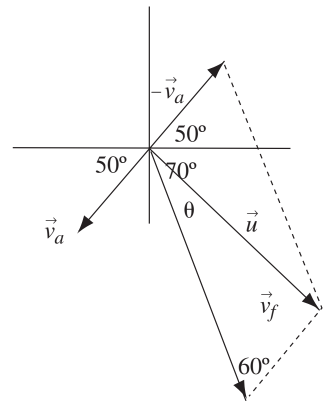
Therefore $4\cos\theta = 1.5$

$$\cos\theta = \frac{1.5}{4}$$

$$\theta \cong 67.976$$

The boat must steer at an angle of 68° to go directly across. The speed going across the river is 4 sin 68° m/s therefore the time to cross is $\frac{650}{4\sin 68°} \cong 175.29$ s or 2.9 min.

8.



Let $\vec{v}_a$ represent the velocity of the aircraft, $\vec{v}_f$ represent the velocity of the fighter jet, and $\vec{u}$ represent the relative velocity of the fighter jet with respect to the aircraft. Hence $\vec{u} = \vec{v}_f - \vec{v}_a$.

$$\begin{aligned} \text{Now } |\vec{u}|^2 &= 735^2 + 300^2 - 2 \cdot 735 \cdot 300 \cos 60° \\ |\vec{u}| &\cong 640.0976. \end{aligned}$$

Also $\frac{\sin\theta}{300} = \frac{\sin 60°}{|\vec{u}|}$

$$\sin\theta = \frac{300 \sin 60°}{|\vec{u}|}$$

$$\theta \cong 23.95°.$$

The relative velocity of the fighter jet with respect to the aircraft is 640 knots with a direction of *S* 44° *E*.